

Re-formation of Many-Quark Model with the $su(4)$ -Algebraic Structure in the Schwinger Boson Realization

— *Reconsideration in the Original Fermion Space* —

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The energy eigenstates for the $su(4)$ -algebraic model for many-quark system obtained by the present authors in the Schwinger boson space are reconsidered in the original fermion space. Through this task, the structures of the single-quark and the quark-triplet are clarified. Variations of the $su(4)$ -generators which have been adopted by the present authors are discussed.

§1. Introduction

The $su(4)$ -algebraic model for many-quark system may be an attractive model. With the aid of this model, we are able to obtain a schematic understanding of not only the quark-triplet but also the quark-pair phase. In addition, if we are interested in the single-quark phase, this model enables us to investigate this phase including the possibility of its existence. Investigation of many-quark system based on the $su(4)$ -algebra traces back to 1983. In this year, Petry et al. proposed an interesting model which has been called the Bonn model.¹⁾ This model leads us to the quark-triplet which may be identified as “nucleon” in the spherical j - j coupling shell model including Δ -particle. In succession, aiming at realistic application, some works have been reported.²⁾ The Bonn model is a kind of effective models based on the quark-pairing interaction. Therefore, in the frame of the Bonn model, we may expect to describe the quark-pairing phase, namely, color superconducting phase.³⁾ Recently, with the aim of investigating the quark-triplet and the quark-pair phase in a unified scheme, the present authors have been concerned with the Bonn model.^{4), 5)} Naturally, preserving the basic aspects of this model, we modified it so as to be able to correct certain insufficient parts. Our description is based on the Schwinger boson representation proposed by the present authors (M.Y.) with Kuriyama and Kunihiro.⁶⁾ The energy eigenvalues and their eigenstates are exactly obtained and we gave various analysis for the results. However, as will be later mentioned, practically, we did not consider the color-symmetric nature of the model explicitly.

Under the above-mentioned circumstances, very recently, the present authors have presented three papers,⁷⁾ which will be referred to as (I) collectively. In the same idea as that of Refs.4) and 5), the $su(4)$ -algebra was treated in the Schwinger boson

space. Of course, we took care of the correspondence to the original fermion space. Usually, the Lie algebraic approach to many-body theory starts in a chosen minimum weight state. We also follow this idea. The starting form of our $su(4)$ -algebraic model is color-symmetric and involves six types of minimum weight states. The eigenvalues of the Casimir operator are all identical to one another. In these six, we choose a certain one. As a natural consequence, the basic equations obtained under the chosen minimum weight state violate the color-symmetry and the eigenstates of the color-symmetric Hamiltonian as a whole are not color-singlets. In Refs.4) and 5), we did not consider this property explicitly. In (I), we gave a certain reasonable interpretation for this problem and showed that the energy eigenvalues are of the same forms as those obtained in Refs.4) and 5). On the basis of these results, we made detailed analyses for the ground-state energies with positive results. For example, the treatment in (I) gives the following conclusion: In the low and the high density region, the quark-triplet and the quark-pair phase are dominant, respectively. The above is consistent to the common understanding. The description of (I) is based on a specific bilinear form of the quark operators for the $su(4)$ -generators and concrete analysis is performed not in the original fermion space, but in the Schwinger boson space. Therefore, for instance, in the framework of (I), it is an open question to identify the quark-triplet obtained in (I) with the “nucleon” in the spherical j - j coupling shell model which characterizes the Bonn model and, further, the structure of the single-quark operator is not clarified in relation to the original fermion operator.

The first aim of this paper is to transcribe the eigenstates of the Hamiltonian obtained in the Schwinger boson space into the corresponding eigenstates in the original fermion space. The idea for the transcription is simple. As was shown in (I), the eigenstates in the Schwinger boson space are derived by operating certain state-generating operators on the minimum weight states. The state-generating operators are certain functions of the $su(4)$ -generators. In Ref.5), we presented other forms by performing this operation explicitly. Through this process, besides the $su(4)$ -generators as the quark-pair creation operators, we could derive the explicit forms of the single-quark and the quark-triplet creation operators in the Schwinger boson space, namely, the building blocks for constructing the eigenstates. Therefore, it may be enough for our task to find the minimum weight states and the correspondences of the state-generating operators in the original fermion space. Through this transcription, we can learn that the single-quark operators are the quark operator themselves and also the structure of the quark-triplets are clarified. In other words, the building blocks for constructing the eigenstates are given.

The second aim is to present various variations of the $su(4)$ -generators in the original fermion space obtained in (I). These are expressed in terms of the bilinear forms of the quark creation and annihilation operators. Each quark operator is specified by a set of quark numbers for the single-particle state besides the color quantum number. Coefficients of the bilinear forms are determined so as to lead to the $su(4)$ -algebra. Therefore, the coefficients depend on the quantum numbers specifying the single-particle states. Following the choice of the quantum numbers, various $su(4)$ -algebraic models appear. In (I), we treated the following case: The total number of the single-particle states, each of which is specified by one quantum

number, is even. For our idea for describing the $su(4)$ -algebraic model, this case may be very instructive. In this paper, after giving a general form, we will show mainly three variations of the expressions for the $su(4)$ -generators. The first and the second variation are for the hadron and nuclear physics. In particular, the relation between the quark-triplet and “nucleon” in the shell model is discussed. The third is for the atomic physics: The model for a trapped three color atom gas is presented by Errea et al.⁸⁾

In next two sections, some aspects of the model treated in (I) are recapitulated. In §2, mainly, the outline of the model is given both in the fermion space and in its corresponding Schwinger boson space. In §3, the eigenstates of the Hamiltonian are given in the Schwinger boson space. The minimum weight state and the state-generating operators are introduced. Section 4 is devoted to presenting the explicit forms of the eigenstates in the original fermion space. In §5, the general expression of the $su(4)$ -generators is presented in terms of covering the form recapitulated in §2. Further, the eigenstates are given for the general case. In §6, the variations of the model given in §2 are discussed. The last section is devoted to a brief summary. In Appendix, some mathematical formulae are given.

§2. The $su(4)$ -algebraic many-quark model and its Schwinger boson realization

In this section, we will recapitulate some aspects of our model by rearranging the results reported in our several papers,^{4), 5), 7)} mainly in (I). This model is formulated in terms of the generators of the $su(4)$ -algebra constructed by the bilinear forms of the quark operators. The color quantum number is denoted by $i = 1, 2$ and 3 . Each color state has the degeneracy $(2j_s + 1)$. Here, j_s denotes a half-integer. For the present, the degrees of freedom related to isospin are ignored. Therefore, any single-particle state is specified as (i, m) with $m = \pm 1/2, \pm 3/2, \dots, \pm j_s$. It should be noted that the above specification for the single-particle state does not always mean the spherical many-fermion system. In §5, we will reconsider this statement. Creation and annihilation operators are denoted as \tilde{c}_{im}^* and \tilde{c}_{im} , respectively.

We introduce the following fifteen operators which are investigated in (I):

$$\tilde{S}^1 = \sum_m \tilde{c}_{2m}^* \tilde{c}_{3\bar{m}}^* , \quad \tilde{S}^2 = \sum_m \tilde{c}_{3m}^* \tilde{c}_{1\bar{m}}^* , \quad \tilde{S}^3 = \sum_m \tilde{c}_{1m}^* \tilde{c}_{2\bar{m}}^* ,$$

$$\tilde{S}_1 = (\tilde{S}^1)^* , \quad \tilde{S}_2 = (\tilde{S}^2)^* , \quad \tilde{S}_3 = (\tilde{S}^3)^* , \quad (2.1a)$$

$$\tilde{S}_1^2 = - \sum_m \tilde{c}_{2m}^* \tilde{c}_{1m} , \quad \tilde{S}_1^3 = - \sum_m \tilde{c}_{3m}^* \tilde{c}_{1m} , \quad \tilde{S}_2^3 = - \sum_m \tilde{c}_{3m}^* \tilde{c}_{2m} ,$$

$$\tilde{S}_2^1 = (\tilde{S}_1^2)^* , \quad \tilde{S}_3^1 = (\tilde{S}_1^3)^* , \quad \tilde{S}_3^2 = (\tilde{S}_2^3)^* , \quad (2.1b)$$

$$\tilde{S}_1^1 = \sum_m (\tilde{c}_{2m}^* \tilde{c}_{2m} + \tilde{c}_{3m}^* \tilde{c}_{3m} - 1) , \quad \tilde{S}_2^2 = \sum_m (\tilde{c}_{3m}^* \tilde{c}_{3m} + \tilde{c}_{1m}^* \tilde{c}_{1m} - 1) ,$$

$$\tilde{S}_3^3 = \sum_m (\tilde{c}_{1m}^* \tilde{c}_{1m} + \tilde{c}_{2m}^* \tilde{c}_{2m} - 1) . \quad (2.1c)$$

Here, we use the notation \tilde{m} for $-m$. The operators (2.1) form the $su(4)$ -algebra:

$$\begin{aligned} [\tilde{S}^i, \tilde{S}^j] &= 0, & [\tilde{S}^i, \tilde{S}_j] &= \tilde{S}_i^j, \\ [\tilde{S}_i^j, \tilde{S}^k] &= \delta_{ij}\tilde{S}^k + \delta_{jk}\tilde{S}^i, & [\tilde{S}_i^j, \tilde{S}_l^k] &= \delta_{jl}\tilde{S}_i^k - \delta_{ik}\tilde{S}_l^j, \end{aligned} \quad (2.2a)$$

The Casimir operator \tilde{P}^2 is expressed as

$$\begin{aligned} \tilde{P}^2 &= \sum_i \left(\tilde{S}_i \tilde{S}^i + \tilde{S}^i \tilde{S}_i \right) + \sum_{i \neq j} \tilde{S}_j^i \tilde{S}_i^j \\ &+ \frac{1}{4} \left[\left(\tilde{S}_2^2 - \tilde{S}_3^3 \right)^2 + \left(\tilde{S}_3^3 - \tilde{S}_1^1 \right)^2 + \left(\tilde{S}_1^1 - \tilde{S}_2^2 \right)^2 \right]. \end{aligned} \quad (2.2b)$$

As a subalgebra, the $su(4)$ -algebra contains the $su(3)$ -algebra. For example, the following set forms the $su(3)$ -algebra:

$$\tilde{S}_1^2, \tilde{S}_2^1, \tilde{S}_1^3, \tilde{S}_3^1, \tilde{S}_2^3, \tilde{S}_3^2, \frac{1}{2} \left(\tilde{S}_2^2 - \tilde{S}_3^3 \right), \tilde{S}_1^1 - \frac{1}{2} \left(\tilde{S}_2^2 + \tilde{S}_3^3 \right). \quad (2.3a)$$

The Casimir operator \tilde{Q}^2 is given in the form

$$\begin{aligned} \tilde{Q}^2 &= \sum_{i \neq j} \tilde{S}_j^i \tilde{S}_i^j + \frac{1}{3} \left[\left(\tilde{S}_2^2 - \tilde{S}_3^3 \right)^2 + \left(\tilde{S}_3^3 - \tilde{S}_1^1 \right)^2 + \left(\tilde{S}_1^1 - \tilde{S}_2^2 \right)^2 \right] \\ &= \sum_{i \neq j} \tilde{S}_j^i \tilde{S}_i^j + 2 \left[\frac{1}{2} \left(\tilde{S}_2^2 - \tilde{S}_3^3 \right) \right]^2 + \frac{2}{3} \left[\tilde{S}_1^1 - \frac{1}{2} \left(\tilde{S}_2^2 + \tilde{S}_3^3 \right) \right]^2. \end{aligned} \quad (2.3b)$$

We can see that \tilde{P}^2 and \tilde{Q}^2 are color-symmetric. By permutations from (1, 2, 3) for the color quantum numbers to others, for example, such as (2, 3, 1), we have six cases including the identical permutation. In order to formulate the present model in terms of the color-symmetric form, it may be necessary to treat the above six cases on an equal footing. It was stressed in (I). At the moment, our discussion is restricted to the case (2.3a). In addition, the operator, which is linearly independent of $(\tilde{S}_2^2 - \tilde{S}_3^3)/2$ and $\tilde{S}_1^1 - (\tilde{S}_2^2 + \tilde{S}_3^3)/2$, is introduced:

$$\tilde{P}_0 = \frac{1}{2} \left(\tilde{S}_1^1 + \tilde{S}_2^2 + \tilde{S}_3^3 \right). \quad (2.4)$$

The operator \tilde{P}_0 is color-symmetric and satisfies

$$[\tilde{S}_i^j, \tilde{P}_0] = 0. \quad (i, j = 1, 2, 3) \quad (2.5)$$

The Hamiltonian adopted in this model is expressed in the form

$$\tilde{H}_m = \tilde{H} + \tilde{\chi} \tilde{Q}^2, \quad \tilde{H} = - \sum_i \tilde{S}^i \tilde{S}_i. \quad (2.6)$$

If $\tilde{\chi} = 0$, \tilde{H}_m reduces to \tilde{H} . It is similar to the form originally adopted in the Bonn model. The Hamiltonian \tilde{H} is characterized by the relation

$$[\tilde{S}_i^j, \tilde{H}] = 0. \quad (2.7)$$

In order to explain certain properties of a many-quark system which cannot be explained in the frame of \tilde{H} , the present authors modified \tilde{H} to \tilde{H}_m by adding the term $\tilde{\chi}\tilde{Q}^2$ to \tilde{H} in (I). We are interested in the change of the energy eigenvalues of \tilde{H}_m from those of \tilde{H} without any change for the energy eigenstates. Therefore, we require the condition

$$[\tilde{S}_i^j, \tilde{\chi}] = 0, \quad \text{i.e.} \quad [\tilde{S}_i^j, \tilde{H}_m] = 0. \quad (2.8)$$

The above is the outline of the framework of our model.

In (I), we have investigated in detail the above $su(4)$ -model in the framework of the Schwinger boson realization. By introducing eight kinds of boson operators (\hat{a}, \hat{a}^*) , (\hat{b}, \hat{b}^*) , (\hat{a}_i, \hat{a}_i^*) and (\hat{b}_i, \hat{b}_i^*) ($i = 1, 2, 3$), the $su(4)$ -algebra can be formulated as follows:

$$\begin{aligned} \tilde{S}^i &\rightarrow \hat{S}^i = \hat{a}_i^* \hat{b} - \hat{a}^* \hat{b}_i, & \tilde{S}_i &\rightarrow \hat{S}_i = \hat{b}^* \hat{a}_i - \hat{b}_i^* \hat{a}, \\ \tilde{S}_i^j &\rightarrow \hat{S}_i^j = (\hat{a}_i^* \hat{a}_j - \hat{b}_j^* \hat{b}_i) + \delta_{ij}(\hat{a}^* \hat{a} - \hat{b}^* \hat{b}). \end{aligned} \quad (2.9)$$

Associating with the above $su(4)$ -algebra, we can define the $su(1,1)$ -algebra:

$$\begin{aligned} \hat{T}_+ &= \hat{a}^* \hat{b}^* + \sum_i \hat{a}_i^* \hat{b}_i^*, & \hat{T}_- &= \hat{b} \hat{a} + \sum_i \hat{b}_i \hat{a}_i, \\ \hat{T}_0 &= \frac{1}{2}(\hat{a}^* \hat{a} + \hat{b}^* \hat{b}) + \frac{1}{2} \sum_i (\hat{a}_i^* \hat{a}_i + \hat{b}_i^* \hat{b}_i) + 2. \end{aligned} \quad (2.10)$$

The commutation relations for $\hat{T}_{\pm,0}$ and the Casimir operator \hat{T}^2 are given in the form

$$[\hat{T}_+, \hat{T}_-] = -2\hat{T}_0, \quad [\hat{T}_0, \hat{T}_{\pm}] = \pm\hat{T}_{\pm}, \quad (2.11a)$$

$$\hat{T}^2 = -\frac{1}{2}(\hat{T}_- \hat{T}_+ + \hat{T}_+ \hat{T}_-) + \hat{T}_0^2. \quad (2.11b)$$

It may be important to see the relation

$$[\text{any of } (\hat{T}_{\pm,0}), \text{any of } (\hat{S}^i, \hat{S}_i, \hat{S}_i^j)] = 0. \quad (2.12)$$

By replacing \tilde{S}^i etc. and $\tilde{\chi}$ in \tilde{H}_m with \hat{S}^i etc. and $\hat{\chi}$, respectively, we obtain the Hamiltonian in the Schwinger boson space, which is denoted as \hat{H}_m . Of course, if $\tilde{\chi}$ is c -number, i.e., $\tilde{\chi} = \chi$, $\hat{\chi}$ should be also c -number, i.e., $\hat{\chi} = \chi$ and the relation (2.12) leads us to

$$[\hat{T}_{\pm,0}, \hat{H}_m] = 0. \quad (2.13)$$

As a concrete example, we have shown the form of $\hat{\chi}$ as a q -number in (I): $\hat{\chi}$ is a certain function of \hat{T}^2 and \hat{P}_0 . The operator \hat{P}_0 is obtained by replacing \tilde{S}_i^i with \hat{S}_i^i ($i = 1, 2, 3$) in \tilde{P}_0 defined in the relation (2.4). In this case, we have

$$[\hat{T}_{\pm,0}, \hat{\chi}] = 0. \quad (2.14)$$

Then, under this condition, the relation (2.13) is realized. Since $\hat{\chi}$ is regarded as a function of \hat{T}^2 and \hat{P}_0 , we have

$$[\hat{S}_i^j, \hat{\chi}] = 0, \quad \text{i.e.,} \quad [\hat{S}_i^j, \hat{H}_m] = 0. \quad (2.15)$$

Under an appropriate choice of $\hat{\chi}$, our model interprets one of the important features of many-quark system: in the low and in the high density region, the quark-triplet and the quark-pair phase are dominant, respectively.

The above is an outline of our model in the Schwinger boson realization. Of course, it has been already reported in several papers^{4),5)} including (I). In this connection, we must comment the following: The form (2.9) is by no means unique. For example, the simplest case is presented by the form $\hat{S}^i = \hat{a}_i^* \hat{b}$, $\hat{S}_i = \hat{b}^* \hat{a}_i$ and $\hat{S}_i^j = \hat{a}_i^* \hat{a}_j - \delta_{ij} \hat{b}^* \hat{b}$. However, the quark-triplet cannot be treated by this representation.

§3. The energy eigenstates in the Schwinger boson space

Our next task is to recapitulate the energy eigenstates of \hat{H}_m . First, we note that our Schwinger boson space is spanned by eight kinds of boson operators and, then, the orthogonal set is specified by eight quantum numbers. The relations (2.13) and (2.14) suggest that, in order to get the energy eigenvalues, it may be enough to search the minimum weight state $|M_1\rangle$ for the $su(1,1)$ - and the $su(3)$ -algebra:

$$\begin{aligned} \hat{T}_- |M_1\rangle &= 0, & \hat{T}_0 |M_1\rangle &= (\sigma + 2) |M_1\rangle, \\ \hat{S}_2^1 |M_1\rangle &= \hat{S}_3^1 |M_1\rangle = \hat{S}_3^2 |M_1\rangle = 0, \end{aligned} \quad (3.1a)$$

$$\begin{aligned} \frac{1}{2} (\hat{S}_2^2 - \hat{S}_3^3) |M_1\rangle &= -\lambda |M_1\rangle, \\ \left[\hat{S}_1^1 - \frac{1}{2} (\hat{S}_2^2 + \hat{S}_3^3) \right] |M_1\rangle &= -(2(\sigma - \sigma_0) + (\lambda - 2\rho)) |M_1\rangle, \end{aligned} \quad (3.1b)$$

$$\hat{P}_0 |M_1\rangle = -(4(\lambda + \rho) - (\sigma + 2\sigma_0)) |M_1\rangle. \quad (3.1c)$$

Obviously, $|M_1\rangle$ is expressed in terms of the four quantum numbers: λ , ρ , σ_0 and σ . In (I), as the state $|M_1\rangle$ satisfying the relation (3.1), we presented the form

$$|M_1\rangle = \|\lambda\rho\sigma_0\sigma\rangle = (\hat{S}^3)^{2\lambda} (\hat{S}^4(1))^{2\rho} (\hat{b}_1^*)^{2(\sigma-\sigma_0)} (\hat{b}^*)^{2\sigma_0} |0\rangle, \quad (3.2)$$

$$\hat{S}^4(1) = -\hat{S}^1 \left(\hat{S}_1^1 - \frac{1}{2} (\hat{S}_2^2 + \hat{S}_3^3) \right) - \hat{S}^2 \hat{S}_1^2 - \hat{S}^3 \hat{S}_1^3. \quad (3.3)$$

Here, in (I), we used \hat{S}^4 which is identical to $-\hat{S}^4(1)$. On the other hand, we presented another form

$$|M_1\rangle = \|lsrw\rangle = (\hat{S}^3)^{2l} (\hat{q}^1)^{2s} (\hat{B}^*)^{2r} (\hat{b}^*)^{2w} |0\rangle. \quad (3.4)$$

It has been proved in Ref.5) that both are equivalent to each other through the relation

$$l = \lambda, \quad s = \sigma - \sigma_0 - \rho, \quad r = \rho, \quad w = \sigma. \quad (3.5)$$

The operators \hat{q}^k and \hat{B}^* are defined as

$$\hat{q}^k = \hat{b}_k^* \hat{b} - \hat{a}^* \hat{a}_k, \quad \hat{B}^* = \sum_{i=1}^3 \hat{S}^i \hat{q}^i. \quad (3.6)$$

The operators \hat{S}^3 , \hat{q}^1 and \hat{B}^* indicate the creation operators of the quark-pair, the single-quark and the quark-triplet, respectively, that is, the building blocks for the state $\|l s r w\rangle$. They carry two, one and three quarks, respectively. Further, we have the relation

$$[\hat{S}_i^j, \hat{S}^k] = \delta_{ij} \hat{S}^k + \delta_{jk} \hat{S}^i, \quad (3.7a)$$

$$[\hat{S}_i^j, \hat{q}^k] = \delta_{ij} \hat{q}^k - \delta_{ik} \hat{q}^j, \quad (3.7b)$$

$$[\hat{S}_i^j, \hat{B}^*] = 2\delta_{ij} \hat{B}^*. \quad (3.7c)$$

For the sake of the terms $\delta_{jk} \hat{S}^i$ and $\delta_{ik} \hat{q}^j$, \hat{S}^3 and \hat{q}^1 are not color-singlet, but \hat{B}^* is color-singlet. Therefore, we conclude that in spite of the eigenstate of \hat{H}_m , the state (3.4) is not color-singlet.

Up to the present, our treatment was based on the $su(3)$ -algebra shown in the form (2.3). In §2, we mentioned that in order to guarantee the color-singlet property, we must take into account the other five forms obtained from the form (2.3) by the permutation for (1, 2, 3). It can be seen that the state $\|l s r w\rangle$ depends on the color quantum number 1 and 3 and, then, we use the notation $\|123; s l r w\rangle$ for $\|l s r w\rangle$. We have two forms $\|231; s l r w\rangle$ and $\|312; s l r w\rangle$. Actually, the above three forms may give the idea of treating our model in the color-symmetric form. In this case, we can find the state $\|c s; s l r w\rangle$ in the form

$$\|c s; s l r w\rangle = \frac{1}{\sqrt{3}} (\|123; s l r w\rangle + \|231; s l r w\rangle + \|312; s l r w\rangle). \quad (3.8)$$

Of course, $\|123; s l r w\rangle$ etc. are normalized. The state (3.8) gives us the relations

$$\langle c s; s l r w | \hat{S}_i^j | c s; s l r w \rangle = 0 \quad \text{for } i \neq j, \quad (3.9)$$

$$\begin{aligned} \langle c s; s l r w | \hat{S}_1^1 | c s; s l r w \rangle &= \langle c s; s l r w | \hat{S}_2^2 | c s; s l r w \rangle = \langle c s; s l r w | \hat{S}_3^3 | c s; s l r w \rangle \\ &= \frac{4}{3}(s + 2l) + 2(2r - w). \end{aligned} \quad (3.10)$$

With the use of the relation (2.1c) with the result (3.10), we can show the result

$$\begin{aligned} \langle c s; s l r w | \hat{N}_1 | c s; s l r w \rangle &= \langle c s; s l r w | \hat{N}_2 | c s; s l r w \rangle = \langle c s; s l r w | \hat{N}_3 | c s; s l r w \rangle \\ &= \frac{N}{3}. \end{aligned} \quad (3.11)$$

Here, \hat{N}_i denotes the quark-number operator of the color i , in the Schwinger boson realization:

$$\begin{aligned} \hat{N}_i &= \frac{1}{2}(2j_s + 1) + \frac{1}{2} \sum_j \hat{S}_j^j - \hat{S}_i^i \\ &= \frac{1}{2}(2j_s + 1) + \frac{1}{2}(\hat{a}^* \hat{a} - \hat{b}^* \hat{b}) + \frac{1}{2} \sum_j (\hat{a}_j^* \hat{a}_j - \hat{b}_j^* \hat{b}_j) - (\hat{a}_i^* \hat{a}_i - \hat{b}_i^* \hat{b}_i). \end{aligned} \quad (3.12)$$

On the average, the color-singlet property of the state $\|cs; slrw\rangle$ is guaranteed. The above is the outline of the energy eigenstates of our model in the Schwinger boson space which has been already reported in (I).

§4. The energy eigenstates in the original fermion space

The main aim of this section is to investigate the explicit form of the energy eigenstates in the original fermion space. If we are interested only in obtaining the energy eigenvalues, it may be enough to treat the present model in the Schwinger boson realization. In fact, we have presented various aspects of the energy eigenvalues in the Schwinger boson space in (I). But, if we are interested also obtaining the energy eigenstates in terms of the constituents, we must turn back to the original fermion space. Through this task, we can complete our investigation of the $su(4)$ -algebraic many-quark model.

First, we reinvestigate the state $\|\lambda\rho\sigma_0\sigma\rangle$. This is obtained by operating with $(\hat{S}^3)^{2\lambda}(\hat{S}^4(1))^{2\rho}$ on the state $|m_1\rangle = \|\lambda = 0, \rho = 0, \sigma_0\sigma\rangle$. In the above sense, \hat{S}^3 and $\hat{S}^4(1)$ can be regarded as the state-generating operators. Then, if we can find $|m_1\rangle$, which corresponds to $|m_1\rangle$, in the fermion space, the state we are looking for is expressed in the form $(\tilde{S}^3)^{2\lambda}(\tilde{S}^4(1))^{2\rho}|m_1\rangle$. Here, $\tilde{S}^4(1)$ is obtained by replacing \hat{S}^1 etc. with \tilde{S}^1 etc. in the form (3.3). The state $|m_1\rangle$ is written down as

$$|m_1\rangle = (\hat{b}_1^*)^{2(\sigma-\sigma_0)}(\hat{b}^*)^{2\sigma_0}|0\rangle. \quad (\sigma \geq \sigma_0) \quad (4.1)$$

Then, we can specify the conditions characterizing $|m_1\rangle$ as follows:

$$\hat{S}_1|m_1\rangle = \hat{S}_2|m_1\rangle = \hat{S}_3|m_1\rangle = 0, \quad (4.2a)$$

$$\hat{S}_2^1|m_1\rangle = \hat{S}_3^1|m_1\rangle = \hat{S}_3^2|m_1\rangle = 0, \quad (4.2b)$$

$$\hat{S}_1^1|m_1\rangle = -2\sigma|m_1\rangle, \quad \hat{S}_2^2|m_1\rangle = -2\sigma_0|m_1\rangle, \quad \hat{S}_3^3|m_1\rangle = -2\sigma'_0|m_1\rangle, \quad (4.2c)$$

$$\sigma'_0 = \sigma_0. \quad (4.2d)$$

The above is nothing but the conditions to determine the minimum weight state for the $su(4)$ -algebra in the case $\sigma'_0 = \sigma_0$. We can formulate the Schwinger boson representation for the $\sigma'_0 \neq \sigma_0$. But, this case may be unsuitable for the Schwinger boson representation for our present fermion model. This point has been discussed in Ref.4).

Under the above argument, we set up the following conditions for the minimum weight state $|m_1\rangle$:

$$\tilde{S}_1|m_1\rangle = \tilde{S}_2|m_1\rangle = \tilde{S}_3|m_1\rangle = 0, \quad (4.3a)$$

$$\tilde{S}_2^1|m_1\rangle = \tilde{S}_3^1|m_1\rangle = \tilde{S}_3^2|m_1\rangle = 0, \quad (4.3b)$$

$$\tilde{S}_1^1|m_1\rangle = -2\sigma|m_1\rangle, \quad \tilde{S}_2^2|m_1\rangle = -2\sigma_0|m_1\rangle, \quad \tilde{S}_3^3|m_1\rangle = -2\sigma'_0|m_1\rangle, \quad (4.3c)$$

$$\sigma'_0 = \sigma_0. \quad (4.3d)$$

For the convenience of later discussion, we will use n_0 and n , which are the eigenvalues of the quark-number operators \tilde{N}_1 , \tilde{N}_2 and \tilde{N}_3 for $|m_1\rangle$:

$$\tilde{N}_1|m_1\rangle = n|m_1\rangle, \quad \tilde{N}_2|m_1\rangle = \tilde{N}_3|m_1\rangle = n_0|m_1\rangle. \quad (4.4)$$

The relation between (σ_0, σ) and (n_0, n) is given by

$$\sigma_0 = \frac{1}{2}(2j_s + 1) - \frac{1}{2}(n_0 + n) , \quad \sigma = \frac{1}{2}(2j_s + 1) - n_0 . \quad (4.5)$$

Of course, the above expression is also valid in the Schwinger boson realization. The form (3.1) tells that $\sigma \geq \sigma_0$ and we have

$$n \geq n_0 . \quad (4.6)$$

The relations (4.3)–(4.5) give $|m_1\rangle$ in the following form:

$$|m_1\rangle = \left(\prod_{p=1}^{n-n_0} \tilde{c}_{i\mu_p}^* \right) \left(\prod_{q=1}^{n_0} \tilde{D}_{\mu'_q}^* \right) |0\rangle . \quad (4.7)$$

Here, \tilde{D}_μ^* denotes

$$\tilde{D}_\mu^* = \prod_{i=1}^3 \tilde{c}_{i\mu}^* . \quad (4.8)$$

Any of $(\mu_1, \dots, \mu_{n-n_0}, \mu'_1, \dots, \mu'_{n_0})$ takes the value between $(-j_s)$ to $(+j_s)$ in agreement with the Pauli-principle. Therefore, we have the following state which corresponds to $\|\lambda\rho\sigma_0\sigma\rangle$:

$$\|\lambda\rho\sigma_0\sigma\rangle \rightarrow (\tilde{S}^3)^{2\lambda} (\tilde{S}^4(1))^{2\rho} \left(\prod_{p=1}^{n-n_0} \tilde{c}_{1\mu_p}^* \right) \left(\prod_{q=1}^{n_0} \tilde{D}_{\mu'_q}^* \right) |0\rangle . \quad (4.9)$$

We rewrote the state (3.2) to the form (3.4). As was already mentioned, the form (3.4) is quite suitable for understanding the structure of many-quark system. We will rewrite the state (4.9), following this idea. For this task, it may be enough to rewrite the part $(\tilde{S}^4(1))^{2\rho} (\prod_{p=1}^{n-n_0} \tilde{c}_{1\mu_p}^*) (\prod_{q=1}^{n_0} \tilde{D}_{\mu'_q}^*) |0\rangle$ to the form suitable for our discussion. For this rewriting, we introduce the operator \tilde{B}_μ^* in the form

$$\tilde{B}_\mu^* = [\tilde{S}^4(1) , \tilde{c}_{1\mu}^*] . \quad (4.10)$$

With the use of the explicit form of $\tilde{S}^4(1)$, \tilde{B}_μ^* is obtained in the following form:

$$\tilde{B}_\mu^* = \sum_{i=1}^3 \tilde{S}^i \tilde{c}_{i\mu}^* . \quad (4.11)$$

The form (4.11) tells that, in spite of the commutator $[\tilde{S}^4(1) , \tilde{c}_{1\mu}^*]$ which is related to the color $i = 1$, the result $\sum_i \tilde{S}^i \tilde{c}_{i\mu}^*$ is color-symmetric. The operator \tilde{D}_μ^* given in the relation (4.8) is also color-symmetric. The operators \tilde{B}_μ^* and \tilde{D}_μ^* carry three quarks and satisfy the relations

$$[\tilde{S}^4(1) , \tilde{B}_\mu^*] = 0 , \quad (4.12a)$$

$$[\tilde{S}^4(1) , \tilde{D}_\mu^*] = 0 . \quad (4.12b)$$

The definition (4.11) supports the following anti-commutation relation:

$$\{ \tilde{c}_{1\mu}^* , \tilde{B}_{\mu'}^* \} = 0 , \quad \{ \tilde{B}_\mu^* , \tilde{B}_{\mu'}^* \} = 0 . \quad (4.12c)$$

Further, we have

$$\tilde{S}^4(1)|0\rangle = 0 . \quad (4.13)$$

If we notice the relations (4.12a) and (4.13), we have

$$\begin{aligned} & \left(\tilde{S}^4(1) \right)^{2\rho} \left(\prod_{p=1}^{n-n_0} \tilde{c}_{1\mu_p}^* \right) \left(\prod_{q=1}^{n_0} \tilde{D}_{\mu'_q}^* \right) |0\rangle \\ &= \left(\tilde{S}^4(1) \right)^{2\rho} \left(\prod_{p=1}^{n-n_0} \tilde{c}_{1\mu_p}^* \right) \left(\prod_{q=1}^{n_0} \tilde{D}_{\mu'_q}^* \right) |0\rangle . \end{aligned} \quad (4.14)$$

Here, $(\tilde{S}^4(1))^{2\rho}(\prod_{p=1}^{n-n_0} \tilde{c}_{1\mu_p}^*)$ denotes the multi-commutator defined in the relation (A.1). Let the quantities appearing in Appendix read the following:

$$\tilde{O} \rightarrow \tilde{S}^4(1) , \quad \tilde{c}_p^* \rightarrow \tilde{c}_{1\mu_p}^* , \quad \tilde{B}_p^* \rightarrow \tilde{B}_{\mu_p}^* , \quad L \rightarrow n - n_0 , \quad M = 2\rho . \quad (4.15)$$

Then, the relation (4.14) can be expressed as

$$\begin{aligned} & \left(\tilde{S}^4(1) \right)^{2\rho} \left(\prod_{p=1}^{n-n_0} \tilde{c}_{1\mu_p}^* \right) \left(\prod_{q=1}^{n_0} \tilde{D}_{\mu'_q}^* \right) |0\rangle \\ &= \frac{1}{(n - n_0 - 2\rho)!} \sum_P (-)^P P \left(\prod_{p=2\rho+1}^{n-n_0} \tilde{c}_{1\mu_p}^* \prod_{p=1}^{2\rho} \tilde{B}_{\mu_p}^* \right) \cdot \left(\prod_{q=1}^{n_0} \tilde{D}_{\mu'_q}^* \right) |0\rangle . \end{aligned} \quad (4.16)$$

Finally, we obtain the following result:

$$\|\lambda\rho\sigma_0\sigma\rangle \rightarrow \frac{1}{(n - n_0 - 2\rho)!} \left(\tilde{S}^3 \right)^{2\lambda} \sum_P (-)^P P \left(\prod_{p=2\rho+1}^{n-n_0} \tilde{c}_{1\mu_p}^* \prod_{p=1}^{2\rho} \tilde{B}_{\mu_p}^* \right) \cdot \left(\prod_{q=1}^{n_0} \tilde{D}_{\mu'_q}^* \right) |0\rangle \quad (4.17)$$

Obviously, the state (4.17) depends on the colors 3 and 1 and it is not color-symmetric. Therefore, with the help of the permutation, the state (4.17) must be symmetrized by the same procedure as the one described in the result (3.8). Then, we obtain the same results as those in (3.8)–(3.11).

Finally, we consider the building blocks of the color-symmetrized version of the state (4.17): \tilde{S}^k , \tilde{c}_{km}^* , \tilde{B}_m^* and \tilde{D}_m^* . These operators satisfy the relations

$$[\tilde{S}_i^j , \tilde{S}^k] = \delta_{ij} \tilde{S}^k + \delta_{jk} \tilde{S}^i , \quad (4.18a)$$

$$[\tilde{S}_i^j , \tilde{c}_{km}^*] = \delta_{ij} \tilde{c}_{km}^* - \delta_{ik} \tilde{c}_{jm}^* , \quad (4.18b)$$

$$[\tilde{S}_i^j , \tilde{B}_m^*] = 2\delta_{ij}\tilde{B}_m^* , \quad (4.19a)$$

$$[\tilde{S}_i^j , \tilde{D}_m^*] = 2\delta_{ij}\tilde{D}_m^* . \quad (4.19b)$$

From the definition of \tilde{S}^k and \tilde{c}_{km}^* and the relation (4.18), we see that these operators create a quark-pair and a single-quark, respectively, and not color-singlet operators. On the other hand, \tilde{B}_m^* and \tilde{D}_m^* create quark-triplet and they are color-singlets. But, concerning their role in the present model, both are different from one another. The minimum weight state is given by the relation (4.7), which consist of the single-quark and the quark-triplet operators \tilde{c}_{1m}^* and \tilde{D}_m^* . The operator \tilde{c}_{1m}^* is transformed through $\tilde{S}^4(1)$, but \tilde{D}_m^* is not affected by $\tilde{S}^4(1)$. This can be seen in the relations (4.10) and (4.12b). Therefore, if the system under investigation can be treated in the framework of a single irreducible representation, the minimum weight state is unchanged and \tilde{B}_m^* describes the system, alone. If the description of the system requires at least two irreducible representations ($n'_0 \neq n_0$), not only \tilde{B}_m^* but also \tilde{D}_m^* are needed for the description. It may be interesting to compare the relations (3.7) and (4.18) with each other. The operators \tilde{S}^k and \tilde{c}_{km}^* correspond to \hat{S}^k and \hat{q}^k , respectively. Judging from the above-mentioned role, \tilde{B}_m^* corresponds to \hat{B}^* . In the Schwinger boson realization, we cannot find any operator which corresponds to \tilde{D}_m^* . The part $(\hat{b}^*)^{2\sigma_0}|0\rangle$ in the state (3.2) does not change under the operation of $\hat{S}^4(1)$ given by (3.3) on the state $(\hat{b}^*)^{2\sigma_0}|0\rangle$. Further, we have $\hat{N}_i(\hat{b}^*)^{2\sigma_0}|0\rangle = n_0(\hat{b}^*)^{2\sigma_0}|0\rangle$. Therefore, $(\hat{b}^*)^{2\sigma_0}|0\rangle$ plays the same role as the state $\prod_{q=1}^{n_0} \tilde{D}_{\mu'_q}^*|0\rangle$.

§5. General form of the $su(4)$ -algebraic many-quark model

With the aim of investigating the variations of the expression (2.1), we formulate the $su(4)$ -algebraic model in a rather general scheme. In this section, we treat many-quark system confined in one single-particle level which consists of $2\Omega_a$ single-particle states in each color. Each single-particle state is specified by a quantum number α . As for α , its range of values is given by

$$\alpha = \pm\frac{1}{2} , \pm\frac{3}{2} , \dots , \pm\left(\Omega_a - \frac{1}{2}\right) , \quad (2\Omega_a : \text{even number}) \quad (5.1a)$$

$$\alpha = 0 , \pm 1 , \pm 2 , \dots , \pm\left(\Omega_a - \frac{1}{2}\right) , \quad (2\Omega_a : \text{odd number}) \quad (5.1b)$$

The relation (5.1) tells that, except $\alpha = 0$ in the relation (5.1b), as a partner of α , we can choose another single-particle, $-\alpha$, which is denoted as $\tilde{\alpha}$. In the case $\alpha = 0$, the partner of α , $\tilde{\alpha}$, is the state α it-self, i.e., $\tilde{\alpha} = \alpha$.

Under the above arrangement and with the use of real function $a(\alpha)$, we define the following fifteen operators:

$$\begin{aligned} \tilde{S}^1 &= \sum_{\alpha} a(\alpha) \tilde{c}_{2\alpha}^* \tilde{c}_{3\tilde{\alpha}}^* , & \tilde{S}^2 &= \sum_{\alpha} a(\alpha) \tilde{c}_{3\alpha}^* \tilde{c}_{1\tilde{\alpha}}^* , & \tilde{S}^3 &= \sum_{\alpha} a(\alpha) \tilde{c}_{1\alpha}^* \tilde{c}_{2\tilde{\alpha}}^* , \\ \tilde{S}_1 &= (\tilde{S}^1)^* , & \tilde{S}_2 &= (\tilde{S}^2)^* , & \tilde{S}_3 &= (\tilde{S}^3)^* , \end{aligned} \quad (5.2a)$$

$$\begin{aligned} \tilde{S}_1^2 &= -\sum_{\alpha} \tilde{c}_{2\alpha}^* \tilde{c}_{1\alpha} , \quad \tilde{S}_1^3 = -\sum_{\alpha} \tilde{c}_{3\alpha}^* \tilde{c}_{1\alpha} , \quad \tilde{S}_2^3 = -\sum_{\alpha} \tilde{c}_{3\alpha}^* \tilde{c}_{2\alpha} , \\ \tilde{S}_2^1 &= (\tilde{S}_1^2)^* , \quad \tilde{S}_3^1 = (\tilde{S}_1^3)^* , \quad \tilde{S}_3^2 = (\tilde{S}_2^3)^* , \end{aligned} \quad (5.2b)$$

$$\begin{aligned} \tilde{S}_1^1 &= \sum_{\alpha} (\tilde{c}_{2\alpha}^* \tilde{c}_{2\alpha} + \tilde{c}_{3\alpha}^* \tilde{c}_{3\alpha} - 1) , \quad \tilde{S}_2^2 = \sum_{\alpha} (\tilde{c}_{3\alpha}^* \tilde{c}_{3\alpha} + \tilde{c}_{1\alpha}^* \tilde{c}_{1\alpha} - 1) , \\ \tilde{S}_3^3 &= \sum_{\alpha} (\tilde{c}_{1\alpha}^* \tilde{c}_{1\alpha} + \tilde{c}_{2\alpha}^* \tilde{c}_{2\alpha} - 1) . \end{aligned} \quad (5.2c)$$

Let $a(\alpha)$ and $a(\tilde{\alpha})$ obey the condition

$$a(\alpha)^2 = 1 , \quad a(\tilde{\alpha}) = a(\alpha) . \quad (5.3a)$$

If $a(\alpha)$ does not satisfy the condition (5.3a), $a(\alpha)$ should vanish:

$$a(\alpha) = 0 . \quad (5.3b)$$

We can prove that the above fifteen operators form the $su(4)$ -algebra, that is, they satisfy the commutation relation (2.2a). If $a(\alpha)$ is changed in the frame of the condition (5.3), we obtain various expressions for the $su(4)$ -algebra.

Even if $a(\alpha)$ is reasonably chosen under physical interpretation, we encounter some cases which do not satisfy the condition (5.3). In these cases, we supplement α with new quantum number β and make the following replacement in the expression (5.2):

$$\alpha \rightarrow \alpha\beta , \quad a(\alpha) \rightarrow a(\alpha)b(\beta) , \quad 2\Omega_a \rightarrow 4\Omega_a\Omega_b . \quad (5.4)$$

We obtain the expression satisfying the $su(4)$ -algebra, if $a(\alpha)b(\beta)$ obey the condition

$$(a(\alpha)b(\beta))^2 = 1 . \quad a(\tilde{\alpha})b(\tilde{\beta}) = a(\alpha)b(\beta) . \quad (5.5)$$

Of course, β is divided into two cases which are similar to those in (5.1a) and (5.1b). If the set of the quantum number (α, β) still does not satisfy the condition (5.5), we proceed with the same task as before and make the following replacement:

$$\alpha \rightarrow \alpha\beta\gamma , \quad a(\alpha) \rightarrow a(\alpha)b(\beta)c(\gamma) , \quad 2\Omega_a \rightarrow 8\Omega_a\Omega_b\Omega_c . \quad (5.6)$$

Of course, $a(\alpha)b(\beta)c(\gamma)$ should satisfy

$$(a(\alpha)b(\beta)c(\gamma))^2 = 1 . \quad a(\tilde{\alpha})b(\tilde{\beta})c(\tilde{\gamma}) = a(\alpha)b(\beta)c(\gamma) . \quad (5.7)$$

Here, γ is divided into two cases which are similar to those in (5.1a) and (5.1b).

The expression of the $su(4)$ -generators (5.2) suggests us that the idea developed in last section is available without any modification. The minimum weight state $|m_1\rangle$ is given in the form

$$|m_1\rangle = \left(\prod_{p=1}^{n-n_0} \tilde{c}_{1\alpha_p}^* \right) \left(\prod_{q=1}^{n_0} \tilde{D}_{\alpha'_q}^* \right) |0\rangle , \quad (5.8)$$

$$\tilde{D}_{\alpha}^* = \prod_{i=1}^3 \tilde{c}_{i\alpha}^* . \quad (5.9)$$

The above forms come from the relations (4.7) and (4.8). The operator \tilde{B}_α^* is given in the form

$$\tilde{B}_\alpha^* = [\tilde{S}^4(1) , \tilde{c}_{1\alpha}^*] = \sum_{i=1}^3 \tilde{S}^i \tilde{c}_{i\alpha}^* . \quad (5.10)$$

Of course, \tilde{B}_α^* and \tilde{D}_α^* satisfy

$$[\tilde{S}^4(1) , \tilde{B}_\alpha^*] = 0 , \quad (5.11a)$$

$$[\tilde{S}^4(1) , \tilde{D}_\alpha^*] = 0 , \quad (5.11b)$$

$$\{ \tilde{c}_{1\alpha}^* , \tilde{B}_{\alpha'}^* \} = 0 , \quad \{ \tilde{B}_\alpha^* , \tilde{B}_{\alpha'}^* \} = 0 . \quad (5.11c)$$

Through the same process as that in §4, we have the following form:

$$\begin{aligned} \|\lambda\rho\sigma_0\sigma\rangle &\rightarrow \frac{1}{(n - n_0 - 2\rho)!} \left(\tilde{S}^3 \right)^{2\lambda} \\ &\times \sum_P (-)^P P \left(\prod_{p=2\rho+1}^{n-n_0} \tilde{c}_{1\alpha_p}^* \prod_{p=1}^{2\rho} \tilde{B}_{\alpha_p}^* \right) \cdot \left(\prod_{q=1}^{n_0} \tilde{D}_{\alpha'_q}^* \right) |0\rangle . \end{aligned} \quad (5.12)$$

It may be not necessary to give the explanation of the notations. In the case where the condition (5.3) does not hold, it may be enough to make the replacement (5.4) or (5.6) to the relations (5.8)–(5.12). Judging from the above treatment, all the relations derived in §4 are valid in general, if the quantum number specifying the single-particle state is changed from m to α . Therefore, the relations (4.18) and (4.19) which characterized the building blocks are valid if m is replaced by α and the argument below the relations (4.18) and (4.19) is valid.

§6. Discussions and some examples

Main task of this section is to apply the general form presented in §5 to certain concrete cases. For this aim, we return to the form given in §2. Let us put $\alpha = m$ and $\Omega_a - 1/2 = j_s$ in the relation (5.1a). Then, the expression (2.1) is obtained by putting $a(\alpha) = 1$ for all α in the relation (5.2) and, certainly, this case satisfies the condition (5.3). In this sense, the expression (2.1) is one of the examples of the general form (5.2). Of course, we have $2\Omega_a = 2j_s + 1$. If we regard j_s as the angular momentum specifying the single-particle level in the spherical j - j coupling shell model, the quantum number m becomes the projection to z -axis. In the case $a(\alpha) = 1$, \tilde{S}^i does not indicate the quark-pair coupled to the angular momentum $J = 0$. If we adopt $a(\alpha) = (-)^{j_s - m}$, \tilde{S}^i becomes the quark-pair coupled to $J = 0$, because of the Clebsch-Gordan coefficient $\langle j_s m j_s - m | 00 \rangle = (-)^{j_s - m} \cdot 1/\sqrt{2j_s + 1}$. However, in this case, we have $a(\tilde{\alpha}) = (-)^{j_s + m} = -a(\alpha)$ and in the frame of $a(\alpha) = (-)^{j_s - m}$, the $su(4)$ -algebra cannot be expected. In order to overcome this discrepancy, inevitably, we must the quantum number β . The spin and the isospin are the properties of the quark except the color. Since j_s is a half-integer, the effect of the spin is already included in α and we have an idea to introduce the isospin as β .

Under the above consideration, let us introduce the isospin to our model. The single-particle state is specified by two quantum number m and τ ($= \pm 1/2$), i.e., $b(\beta) = (-)^{\frac{1}{2}-\tau}$ and $2\Omega_b = 2$ and $a(\alpha)b(\beta)$ is expressed as

$$a(\alpha)b(\beta) = (-)^{j_s-m}(-)^{\frac{1}{2}-\tau}, \quad (a(\alpha)b(\beta))^2 = 1. \quad (6.1)$$

Certainly, we have

$$a(\tilde{\alpha})b(\tilde{\beta}) = (-)^{j_s+m}(-)^{\frac{1}{2}+\tau} = a(\alpha)b(\beta). \quad (6.2)$$

The relation (6.1) tells us that we are considering the quark-pair with $J = 0$ and $T = 0$. By substituting the relation (6.1) into the form (5.2), we obtain the expression of the $su(4)$ -generators in the present case. For example, we have $\tilde{S}^1 = \sum_{m\tau} (-)^{j_s-m}(-)^{\frac{1}{2}-\tau} \tilde{c}_{2m\tau}^* \tilde{c}_{3\tilde{m}\tilde{\tau}}^*$ and $\tilde{S}_1^1 = \sum_{m\tau} (\tilde{c}_{2m\tau}^* \tilde{c}_{2m\tau} + \tilde{c}_{3m\tau}^* \tilde{c}_{3m\tau}) - 2(2j_s+1)$. The quark-triplets can be expressed as

$$\tilde{B}_{m\tau}^* = \sum_{i=1}^3 \tilde{S}^i \tilde{c}_{im\tau}^*, \quad \tilde{D}_{m\tau}^* = \prod_{i=1}^3 \tilde{c}_{im\tau}^*. \quad (6.3)$$

The form (6.3) is nothing but the form proposed in the original Bonn model, in which the former and the latter represent “nucleon” and Δ -excitation. In the introductory part of §5, we showed two cases. The above example is based on the case (5.1a).

Next, we consider the case (5.1b). In parallel with (j_s, m) , we introduce a set of the integers (l_s, μ) . For a given l_s , μ takes the value $\mu = 0, \pm 1, \pm 2, \dots, \pm l_s$ and we put $\alpha = \mu$ and $\Omega_a - 1/2 = l_s$ in the relation (5.1b). In the same idea as that in the case (j_s, m) , we regard (l_s, μ) as the angular momentum (orbital). The function $a(\alpha) = (-)^{l_s-\mu}$ satisfies $a(\tilde{\alpha}) = (-)^{l_s+\mu} = a(\alpha)$ and, then, different from the previous case, we obtain the $su(4)$ -algebra. However, in the frame of (l_s, μ) , only the orbital angular momentum is taken into account. Therefore, for the problem of the quark, we must introduce the degrees of freedom related to the spin and the isospin. The single-particle state is specified by three quantum numbers $\mu, \sigma (= \pm 1/2)$ and $\tau (= \pm 1/2)$, i.e., $b(\beta) = (-)^{\frac{1}{2}-\sigma}$, $c(\gamma) = (-)^{\frac{1}{2}-\tau}$, $2\Omega_b = 2$ and $2\Omega_c = 2$. Then, $a(\alpha)b(\beta)c(\gamma)$ is expressed as

$$a(\alpha)b(\beta)c(\gamma) = (-)^{l_s-\mu}(-)^{\frac{1}{2}-\sigma}(-)^{\frac{1}{2}-\tau}, \quad (a(\alpha)b(\beta)c(\gamma))^2 = 1. \quad (6.4)$$

From the following relation, we can expect the $su(4)$ -algebra:

$$a(\tilde{\alpha})b(\tilde{\beta})c(\tilde{\gamma}) = (-)^{l_s+\mu}(-)^{\frac{1}{2}+\sigma}(-)^{\frac{1}{2}+\tau} = a(\alpha)b(\beta)c(\gamma). \quad (6.5)$$

For instance, we have the relations $\tilde{S}^1 = \sum_{\mu\sigma\tau} (-)^{l_s-\mu}(-)^{\frac{1}{2}-\sigma}(-)^{\frac{1}{2}-\tau} \tilde{c}_{2\mu\sigma\tau}^* \tilde{c}_{3\tilde{\mu}\tilde{\sigma}\tilde{\tau}}^*$ and $\tilde{S}_1^1 = \sum_{\mu\sigma\tau} (\tilde{c}_{2\mu\sigma\tau}^* \tilde{c}_{2\mu\sigma\tau} + \tilde{c}_{3\mu\sigma\tau}^* \tilde{c}_{3\mu\sigma\tau}) - 4(2l_s+1)$. In the same manner as the previous case, the quark-triplets in the present case are given in the form

$$\tilde{B}_{\mu\sigma\tau}^* = \sum_{i=1}^3 \tilde{S}^i \tilde{c}_{i\mu\sigma\tau}^*, \quad \tilde{D}_{\mu\sigma\tau}^* = \prod_{i=1}^3 \tilde{c}_{i\mu\sigma\tau}^*. \quad (6.6)$$

The former and the latter denote “nucleon” and Δ -excitation. In contrast with the case (5.1a), the case (5.1b) is based on the spherical L - S coupling shell model.

As is clear from the above argument, we have two forms for the “nucleon”. The forms (6.3) and (6.6) are based on the spherical j - j coupling and L - S coupling shell models, respectively. In subsequent paper, we will discuss these two forms and some problems related to them.

Concerning the condition (5.1b), we have discussed the case $a(\alpha) = (-)^{l_s - \mu}$. The case $a(\alpha) = 1$ for all α gives us the $su(4)$ -algebra and this case can be regarded as the correspondence of the condition (5.1a) with $a(\alpha) = 1$. As a case which cannot be found in the condition (5.1a), we consider the $a(\alpha) = 1$ with $2\Omega_a = 1$. This is nothing but the case $\alpha = 0$ which consists of $(\tilde{c}_{i0}^*, \tilde{c}_{i0})$ for $i = 1, 2$ and 3 . But, if limited to one single-particle level with $\alpha = 0$, the model is too simple to intend to investigate. Then, we enlarge the number of the single-particle level from one to the plural and each level is specified as ν ($\nu = 0, 1, 2, \dots, \nu_0$). Therefore, the single-particle state is specified by $i\nu$: $(\tilde{c}_{i\nu}^*, \tilde{c}_{i\nu})$.

The $su(4)$ -generators for ν -th single-particle level are expressed as follows:

$$\begin{aligned} \tilde{S}^1(\nu) &= \tilde{c}_{2\nu}^* \tilde{c}_{3\nu}^* , & \tilde{S}^2(\nu) &= \tilde{c}_{3\nu}^* \tilde{c}_{1\nu}^* , & \tilde{S}^3(\nu) &= \tilde{c}_{1\nu}^* \tilde{c}_{2\nu}^* , \\ \tilde{S}_1(\nu) &= (\tilde{S}^1(\nu))^* , & \tilde{S}_2(\nu) &= (\tilde{S}^2(\nu))^* , & \tilde{S}_3(\nu) &= (\tilde{S}^3(\nu))^* , \end{aligned} \quad (6.7a)$$

$$\begin{aligned} \tilde{S}_1^2(\nu) &= -\tilde{c}_{2\nu}^* \tilde{c}_{1\nu} , & \tilde{S}_1^3(\nu) &= -\tilde{c}_{3\nu}^* \tilde{c}_{1\nu} , & \tilde{S}_2^3(\nu) &= -\tilde{c}_{3\nu}^* \tilde{c}_{2\nu} , \\ \tilde{S}_2^1(\nu) &= (\tilde{S}_1^2(\nu))^* , & \tilde{S}_3^1(\nu) &= (\tilde{S}_1^3(\nu))^* , & \tilde{S}_3^2(\nu) &= (\tilde{S}_2^3(\nu))^* , \end{aligned} \quad (6.7b)$$

$$\begin{aligned} \tilde{S}_1^1(\nu) &= \tilde{c}_{2\nu}^* \tilde{c}_{2\nu} + \tilde{c}_{3\nu}^* \tilde{c}_{3\nu} - 1 , & \tilde{S}_2^2(\nu) &= \tilde{c}_{3\nu}^* \tilde{c}_{3\nu} + \tilde{c}_{1\nu}^* \tilde{c}_{1\nu} - 1 , \\ \tilde{S}_3^3(\nu) &= \tilde{c}_{1\nu}^* \tilde{c}_{1\nu} + \tilde{c}_{2\nu}^* \tilde{c}_{2\nu} - 1 . \end{aligned} \quad (6.7c)$$

Further, we have the relation for $\nu' \neq \nu$

$$[\text{any of the } \nu'\text{-th generators} , \text{any of the } \nu\text{-th generators}] = 0 . \quad (6.8)$$

Fermion number operator in the ν -th level, $\tilde{N}(\nu)$, is expressed as

$$\tilde{N}(\nu) = \sum_i \tilde{N}_i(\nu) , \quad \tilde{N}_i(\nu) = \tilde{c}_{i\nu}^* \tilde{c}_{i\nu} = \frac{1}{2} \left(\sum_j \tilde{S}_j^j(\nu) + 1 \right) - \tilde{S}_i^i(\nu) . \quad (6.9a)$$

Fermion number operator in the color i and the total fermion number operator are expressed as

$$\tilde{N}_i = \sum_\nu \tilde{N}_i(\nu) = \tilde{N} - \sum_\nu \tilde{S}_i^i(\nu) - \nu , \quad (6.9b)$$

$$\tilde{N} = \sum_\nu \tilde{N}(\nu) . \quad (6.9c)$$

The orthogonal set in the ν -th subspace is given as follows:

$$\begin{aligned} |0\rangle , & \quad \tilde{c}_{2\nu}^* \tilde{c}_{3\nu}^* |0\rangle , \quad \tilde{c}_{3\nu}^* \tilde{c}_{1\nu}^* |0\rangle , \quad \tilde{c}_{1\nu}^* \tilde{c}_{2\nu}^* |0\rangle , \\ \tilde{c}_{1\nu}^* |0\rangle , & \quad \tilde{c}_{2\nu}^* |0\rangle , \quad \tilde{c}_{3\nu}^* |0\rangle , \quad \tilde{c}_{1\nu}^* \tilde{c}_{2\nu}^* \tilde{c}_{3\nu}^* |0\rangle . \end{aligned} \quad (6.10)$$

The vacuum $|0\rangle$ is common to all subspace.

The Hamiltonian adopted in this model, for example, is expressed in the form

$$\tilde{H} = \sum_{\nu} \epsilon_{\nu} \tilde{N}(\nu) - G \sum_i \left(\sum_{\nu} \tilde{S}^i(\nu) \right) \left(\sum_{\nu} \tilde{S}_i(\nu) \right) . \quad (G > 0) \quad (6.11)$$

Here, ϵ_{ν} and G denote the single-particle energy of the ν -th level and the interaction strength, respectively. The order of the level is given as $\epsilon_1 \leq \epsilon_2 \leq \dots \leq \epsilon_{\nu}$. This model is a kind of $su(4) \otimes su(4) \otimes \dots \otimes su(4)$ -algebraic model. If $\epsilon_1 = \epsilon_2 = \dots = \epsilon_{\nu}(= \epsilon)$, the Hamiltonian (6.11) reduces to the following:

$$\tilde{H} = \epsilon \tilde{N} - G \sum_i \tilde{S}^i \tilde{S}_i , \quad \tilde{S}^i = \sum_{\nu} \tilde{S}^i(\nu) . \quad (6.12)$$

The Hamiltonian (6.12) may be equivalent to that of the Bonn model. The above model has been investigated by Errea et al.⁸⁾ by the name of the model for a trapped three-color atom gas. If we adopt the Schwinger boson representation, this model may be described in the form much simpler than that by Errea et al. The conventional and familiar many-body technique can be easily applied. Of course, in this case, the Schwinger boson space is constructed by the bosons $(\hat{a}_i(\nu), \hat{a}_i^*(\nu))$, $(\hat{b}_i(\nu), \hat{b}_i^*(\nu))$, $(\hat{a}(\nu), \hat{a}^*(\nu))$ and $(\hat{b}(\nu), \hat{b}^*(\nu))$ for $i = 1, 2, 3$ and $\nu = 0, 1, 2, \dots, \nu_0$.

§7. Summary

In this paper, we have re-formulated the many-quark model with the $su(4)$ -algebraic structure developed in our previous papers (I) and Refs.4) and 5). In those papers, we investigated the modified Bonn quark model by using the Schwinger boson realization in the boson space and constructed the exact eigenstates together with exact eigenvalues for the energy under consideration. The eigenstates were constructed by the use of operators with the number of quarks, 1, 2 and 3, which were interpreted as the single-quark, the quark-pair and the quark-triplet operator, respectively, in the boson space. In this paper, these operators have been transcribed into the quark operators in the original fermion space. Through this transcription, it was clarified that the single-quark operators were the quark operators themselves. Further, the structure of the quark-triplet operators were also clarified, in which the “nucleon” and “ Δ -excitation” were indicated in the original fermion space.

Furthermore, in this paper, it has been shown that, following the choice of the quantum numbers of the original fermion operator besides the color quantum number, we can construct the various $su(4)$ -algebraic models. If the flavor (isospin) quantum number is introduced adding to the total angular momentum j_s in a single-particle level, the Bonn quark model can be obtained in the j - j coupling scheme. When the orbital angular momentum (l_s, μ) , instead of (j_s, m) , is treated together with the quark spin $(1/2, \pm 1/2)$, the Bonn quark model based on the L - S coupling scheme appears. Further, if many single-particle levels are introduced, the $su(4) \otimes su(4) \otimes \dots \otimes su(4)$ -algebraic model can be constructed, which may be known as a model of a trapped three color atom gas in the atomic physics. Thus, it is concluded

that the many-fermion models with the $su(4)$ -algebraic structure may be treated widely in our scheme developed in our papers and re-formulated in this paper.

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Appendix A

—— General formula for deriving the relation (4.16) ——

In this Appendix, we will present a formula for deriving the relation (4.16) in a rather general form. Our present problem is to give an idea how to express the multiple commutator of \tilde{O} for $\tilde{c}_L^* \cdots \tilde{c}_1^*$ which is defined in the following form:

$$(\vec{O})^M(\tilde{c}_L^* \cdots \tilde{c}_1^*) = [\underbrace{\tilde{O}, [\tilde{O}, \cdots, [\tilde{O}, \tilde{c}_L^* \cdots \tilde{c}_1^*] \cdots]}_M]. \quad (\text{A}\cdot 1)$$

The symbol $\tilde{O}\tilde{X}$ denotes the commutator $[\tilde{O}, \tilde{X}]$ and $(\tilde{c}_l^*, \tilde{c}_l; l = 1, 2, \cdots, L)$ is a set of fermion operators. The operator \tilde{O} is a function of $(\tilde{c}_l^*, \tilde{c}_l; l = 1, 2, \cdots, L)$ and we require the following condition:

$$[\tilde{O}, \tilde{B}_l^*] = 0, \quad \{\tilde{c}_l^*, \tilde{B}_{l'}^*\} = 0, \quad \{\tilde{B}_l^*, \tilde{B}_{l'}^*\} = 0. \quad (\text{A}\cdot 2)$$

Here, \tilde{B}_l^* is defined as

$$\tilde{B}_l^* = [\tilde{O}, \tilde{c}_l^*]. \quad (\text{A}\cdot 3)$$

First, we notice the identity

$$\tilde{c}_L^* \cdots \tilde{c}_1^* = (L!)^{-1} \sum_P (-)^P P(\tilde{c}_L^* \cdots \tilde{c}_1^*). \quad (\text{A}\cdot 4)$$

Here, P denotes the permutation

$$P = \begin{pmatrix} 1 & 2 & \cdots & L \\ p_1 & p_2 & \cdots & p_L \end{pmatrix}, \quad (-)^P = \begin{cases} +1 & \text{; even permutation} \\ -1 & \text{; odd permutation} \end{cases} \quad (\text{A}\cdot 5)$$

The simplest case is as follows:

$$\begin{aligned} \vec{O}(\tilde{c}_L^* \cdots \tilde{c}_1^*) &= (L!)^{-1} \sum_P (-)^P P(\vec{O}(\tilde{c}_L^* \cdots \tilde{c}_1^*)) \\ &= (L!)^{-1} \sum_P (-)^P P(\tilde{B}_L^* \tilde{c}_{L-1}^* \cdots \tilde{c}_2^* \tilde{c}_1^* + \tilde{c}_L^* \tilde{B}_{L-1}^* \cdots \tilde{c}_2^* \tilde{c}_1^* + \cdots \\ &\quad + \tilde{c}_L^* \tilde{c}_{L-1}^* \cdots \tilde{B}_2^* \tilde{c}_1^* + \tilde{c}_L^* \tilde{c}_{L-1}^* \cdots \tilde{c}_2^* \tilde{B}_1^*) \\ &= (L!)^{-1} L \sum_P (-)^P P(\tilde{c}_L^* \tilde{c}_{L-1}^* \cdots \tilde{c}_2^* \tilde{B}_1^*) \\ &= ((L-1)!)^{-1} \sum_P (-)^P P(\tilde{c}_L^* \tilde{c}_{L-1}^* \cdots \tilde{c}_2^* \tilde{B}_1^*). \end{aligned} \quad (\text{A}\cdot 6)$$

Of course, we used the relation (A.2) and (A.3). By calculating successively to the higher power for \vec{O} , we obtain the following form:

$$(\vec{O})^M(\tilde{c}_L^* \cdots \tilde{c}_1^*) = ((L-M)!)^{-1} \sum_P (-)^P P(\tilde{c}_L^* \cdots \tilde{c}_{M+1}^* \tilde{B}_M^* \cdots \tilde{B}_1^*) . \quad (\text{A.7})$$

Especially, for the case $M = L$, we have

$$\begin{aligned} (\vec{O})^L(\tilde{c}_L^* \cdots \tilde{c}_1^*) &= \sum_P (-)^P P(\tilde{B}_L^* \cdots \tilde{B}_1^*) \\ &= L! (\tilde{B}_L^* \cdots \tilde{B}_1^*) . \end{aligned} \quad (\text{A.8})$$

We can see that all \tilde{c}_l^* are replaced with the corresponding \tilde{B}_l^* .

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